Analytic Study of Biased Proportional Navigation

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Proportional navigation has proved to be a useful guidance technique in several surface-to-air and air-to-air missile systems for interception of airborne targets. An analytic study of the biased proportional navigation with varying closing speed and maneuvering target is presented in this article. A specific target maneuver is considered that is proportional to the closing rate for effective escape during intercept period. The closed-form solutions are completely derived, and some important characteristics related to the system performance are discussed. The effects of the bias factor and target escape factor on the capture criterion and cumulative velocity increment are investigated in detail. In this scheme, the line-of-sight rate approaches a nonzero value, which is beyond the dead-zone of the system. But intercept still can be achieved with a cost of additional energy, and the capture area will be decreased due to the effect of bias.

Nomenclature

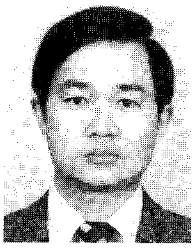
A = $\dot{r}_0/r_0\dot{\theta}_0$
a = acceleration
B = $\dot{\theta}_B/\dot{\theta}_0$
C = proportional constant of target maneuver
D = $-C/\dot{\theta}_B$
E = $-C/\dot{\theta}_0 = BD$
M = zero effort miss
r = distance between interceptor and target
T = time to go
t = time
V = velocity
v = relative velocity between interceptor and target
θ = angle between line of sight and inertial reference line
λ = effective proportional navigation constant

Subscripts
B = bias value
f = final value
M = missile, interceptor
min = minimum value
r = parallel to line of sight
T = target
θ = normal to line of sight
0 = initial value
1 = state at $\dot{\theta} = \dot{\theta}_B$

Superscript
$(\dot{\ })$ = time derivative

I. Introduction

BIASED proportional navigation (biased PN) was discussed early by Murtaugh and Criel,[1] in which the commanded acceleration is biased by a small value of the measured rotational rate of the line of sight (LOS) between the interceptor and its target. In normal proportional navigation, as the LOS rate approaches zero, seeker tracking noise can cause the measured LOS rate to vacillate between positive and negative values. This requires the interceptor to accelerate positive as well as negative, thereby having a severe effect on the control system. To alleviate this effect, the proportional navigation with a bias, in which the maneuver force is set to zero when the absolute LOS rate has been reduced below a specific value, can be employed. With this scheme, the LOS rate is reduced to a nonzero value, but intercept still can be achieved with a cost of additional energy. In this study, the closing speed between the interceptor and its target is assumed to be unchanged and



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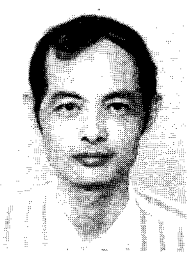

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target maneuver is not considered. Also, the effect of bias on capture area is not discussed. In Refs. 2 and 3, different biased proportional navigation with a constant bias was considered for a maneuvering target.

In this article, the biased proportional navigation developed by Murtaugh will be studied further with varying closing speed and maneuvering target for the case of exoatmospheric flight. Here, a specific target maneuver is considered that is proportional to the closing rate for effective escape during intercept period. A bias factor and a target escape factor are defined, the closed-form solution will be completely derived, and some important characteristics related to the system performance are presented. Also, the induced effects on capture area and cumulative velocity increment will be investigated in detail. The cumulative velocity increment required is related to the propellant mass required in exoatmospheric intercept.

II. Solution for a Nonmaneuvering Target

Consider that a missile of speed V_M is pursuing a nonmaneuvering target with speed V_T under the guidance law of biased proportional navigation, as shown in Fig. 1. Here, the commanded acceleration a_c is given in the direction normal to LOS and its magnitude is

$$a_c = \lambda v_r(\dot\theta - \dot\theta_B) \qquad \text{if} \quad \frac{\dot\theta}{\dot\theta_B} \geq 1$$
$$= 0 \qquad \text{if} \quad \frac{\dot\theta}{\dot\theta_B} < 1 \tag{1}$$

where λ is the effective proportional navigation constant, v_r is the rate of range to go between missile and target, $\dot\theta$ is the angular rate of LOS, and $\dot\theta_B$ is the bias value that is chosen to have the same sign of $\dot\theta$, as depicted in Fig. 2. The relative velocity between missile and target can be written in a polar coordinate as

$$V_M - V_T = v_r e_r + v_\theta e_\theta = \dot r e_r + r\dot\theta e_\theta \tag{2}$$

where r is the range to go between missile and target, and θ is the angle between LOS and inertial reference line. Then the equations of motion are

$$\ddot r - r\dot\theta^2 = 0 \tag{3a}$$

$$r\ddot\theta + 2\dot r\dot\theta = \lambda\dot r(\dot\theta - \dot\theta_B) \qquad \text{if} \quad \frac{\dot\theta}{\dot\theta_B} \geq 1 \tag{3b}$$

$$= 0 \qquad \text{if} \quad \frac{\dot\theta}{\dot\theta_B} < 1 \tag{3c}$$

Now let $\dot\theta_0$ be the initial condition of $\dot\theta$ and $B = \dot\theta_B/\dot\theta_0$ be the bias factor, which is a positive constant. First consider the case of $B \leq 1$, the solutions of Eqs. (3a) and (3b) can be obtained easily by using the results in Ref. 4:

$$\dot\theta = \dot\theta_0\left(\frac{r}{r_0}\right)^{\lambda-2} + \frac{\lambda}{\lambda-2}\dot\theta_B\left[1 - \left(\frac{r}{r_0}\right)^{\lambda-2}\right]$$
$$= \dot\theta_0\left\{\left(\frac{r}{r_0}\right)^{\lambda-2} + \frac{\lambda}{\lambda-2}B\left[1 - \left(\frac{r}{r_0}\right)^{\lambda-2}\right]\right\} \tag{4a}$$

$$\dot r^2 = r_0^2\dot\theta_0^2\left\{\frac{1}{\lambda-1}\left[1 - \frac{\lambda}{\lambda-2}B\right]^2\left(\frac{r}{r_0}\right)^{2\lambda-2}\right.$$
$$\left. + \frac{4}{\lambda-2}B\left[1 - \frac{\lambda}{\lambda-2}B\right]\left(\frac{r}{r_0}\right)^{\lambda} + \frac{\lambda^2}{(\lambda-2)^2}B^2\left(\frac{r}{r_0}\right)^2\right\}$$
$$+ \dot r_0^2 - \frac{r_0^2\dot\theta_0^2}{\lambda-1}(1 + 2B + \lambda B^2) \tag{4b}$$

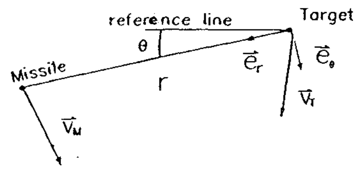

Fig. 1 Planar pursuit geometry.

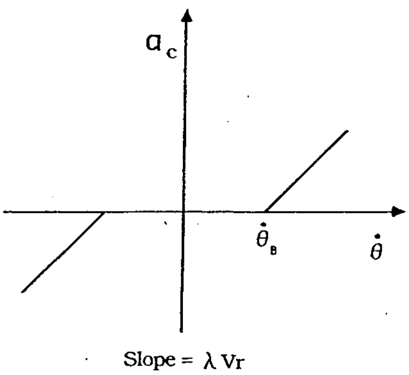

Fig. 2 Biased proportional navigation: a_c vs $\dot\theta$.

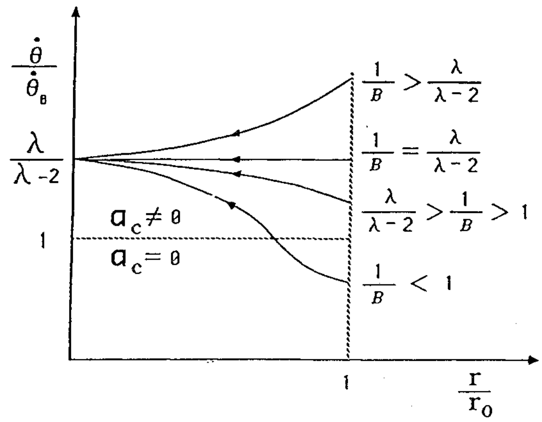

Fig. 3 Different values of B: $\dot\theta$ vs r.

where r_0 and $\dot r_0$ are the initial conditions of r and $\dot r$, respectively. Also we find that, from Eq. (4a), $\dot\theta$ approaches $[\lambda/(\lambda-2)]\dot\theta_B$ at intercept. Figure 3 shows a typical variation between $\dot\theta$ and r for biased proportional navigation, and the closing rate is always monotoneously decreasing during intercept period. The bias also affects the variation of closing rate, as depicted in Fig. 4, and the final rate of range to go at intercept can be derived as

$$\dot r_f = \dot r_0\sqrt{1 - \frac{1 + 2B + \lambda B^2}{A^2(\lambda-1)}} \tag{5}$$

where $A = \dot r_0/r_0\dot\theta_0$, and the final commanded acceleration approaches $[2\lambda/(\lambda-2)]\dot r_f\dot\theta_B$. For the purpose of effective intercept, the following constraint must be satisfied:

$$A^2 > \frac{1 + 2B + \lambda B^2}{\lambda - 1} \tag{6}$$

which can be considered as a criterion of capture capability under the effect of bias. The cumulative velocity increment ΔV required for the missile during intercept period is defined as

$$\Delta V = \int_0^{t_f} |a_c|\, dt \tag{7}$$

Using the results in Eq. (4a), we have

$$\Delta V = \left|\int_{r_0}^{0} \frac{a_c}{\dot r}\,\mathrm{d}r\right| = |r_0\dot\theta_0|\lambda\int_0^1\left[\left(1-\frac{\lambda}{\lambda-2}B\right)\left(\frac{r}{r_0}\right)^{\lambda-2}\right.$$
$$\left.+\frac{2}{\lambda-2}B\right]\mathrm{d}\left(\frac{r}{r_0}\right) = |r_0\dot\theta_0|\frac{\lambda}{\lambda-1}(1+B)$$
$$=\frac{2}{\lambda-1}r_0[|\dot\theta_0|+|\dot\theta_B|] = \Delta V_{\dot\theta_0}+\Delta V_{\dot\theta_B} \tag{8}$$

where

$$\Delta V_{\dot\theta_0} = \frac{\lambda}{\lambda-1}r_0|\dot\theta_0|$$

$$\Delta V_{\dot\theta_B} = \frac{\lambda}{\lambda-1}r_0|\dot\theta_B|$$

Because $\dot\theta_B$ is chosen to have the same sign as $\dot\theta_0$, this ΔV required for the case of biased PN is always greater than that required for the case of normal PN with an additional increment $\Delta V_{\dot\theta_B}$. Now a zero effort miss M is defined as the miss distance that would result if guidance were terminated at a specified time during intercept period. It can be derived as a function of r, v_r, and v_θ (see Appendix A):

$$M = r\sqrt{\frac{v_\theta^2}{v_r^2+v_\theta^2}} = \frac{r}{\sqrt{(v_r^2/v_\theta^2)+1}} \tag{9}$$

Let v_{r_0}, v_{θ_0}, and M_0 be the initial condition of v_r, v_θ, and M, respectively. When $|v_{\theta_0}|$ is much smaller than $|v_{r_0}|$, then the results obtained by Murtaugh and Criel[1]

$$T_0 = \frac{r_0}{v_{r_0}} \tag{10a}$$

$$M_0 = \frac{r_0 v_{\theta_0}}{v_{r_0}} = T_0 v_{\theta_0} \tag{10b}$$

$$\Delta V_{\dot\theta_0} = \frac{\lambda}{\lambda-1}\frac{|M_0|}{T_0} \tag{10c}$$

can be derived from Eqs. (8) and (9), where T_0 is the initial condition of time to go T.

Next we will consider the case of $B>1$. In this case, $\dot\theta$ diverges first to $\dot\theta_B$ with $a_c=0$, where the corresponding value of r is r_1, and then diverges continuously to $[\lambda/(\lambda-2)]\dot\theta_B$ with $a_c\neq 0$ until intercept. That is, we have the following.

1) When $r>r_1$: Because a_c is zero, then the solutions from Eqs. (3a) and (3c) are

$$\dot\theta = \dot\theta_0\left(\frac{r}{r_0}\right)^{-2} \tag{11a}$$

$$\dot r^2 = -r_0^2\dot\theta_0^2\left(\frac{r}{r_0}\right)^{-2} + \dot r_0^2 + r_0^2\dot\theta_0^2 \tag{11b}$$

Let $\dot r_1$ be the value of $\dot r$ at the time when $\dot\theta$ reaches $\dot\theta_B$, then we have

$$\dot\theta_1 = \dot\theta_B = \dot\theta_0 B \tag{12a}$$

$$r_1 = r_0\left(\frac{\dot\theta_0}{\dot\theta_B}\right)^{1/2} = r_0 B^{-1/2} \tag{12b}$$

$$\dot r_1 = \dot r_0\sqrt{1+\frac{1}{A^2}(1-B)} \tag{12c}$$

where B is positive and greater than 1 in this case and $A = \dot r_0/r_0\dot\theta_0$. After r reaches r_1, it decreases continuously with $a_c\neq 0$ as follows.

2) When $r\le r_1$: The solutions here will be similar to those depicted in Eqs. (4). With the initial condition $\dot\theta=\dot\theta_1$, $r=r_1$ and $\dot r=\dot r_1$, we have

$$\dot\theta = \dot\theta_1\left[\frac{\lambda}{\lambda-2}-\frac{2}{\lambda-2}\left(\frac{r}{r_1}\right)^{\lambda-2}\right]$$
$$= \dot\theta_0 B\left[\frac{\lambda}{\lambda-2}-\frac{2}{\lambda-2}\left(\frac{B^{1/2}r}{r_0}\right)^{\lambda-2}\right] \tag{13a}$$

$$\dot r^2 = r_1^2\dot\theta_1^2\left[\frac{4}{(\lambda-1)(\lambda-2)^2}\left(\frac{r}{r_1}\right)^{2\lambda-2} - \frac{8}{(\lambda-2)^2}\left(\frac{r}{r_1}\right)^{\lambda}\right.$$
$$\left.+\frac{\lambda^2}{(\lambda-2)^2}\left(\frac{r}{r_1}\right)^2\right] + \dot r_1^2 - \frac{r_1^2\dot\theta_1^2}{\lambda-1}(3+\lambda)$$
$$= r_0^2\dot\theta_0^2 B\left[\frac{4}{(\lambda-1)(\lambda-2)^2}\left(\frac{B^{1/2}r}{r_0}\right)^{2\lambda-2} - \frac{8}{(\lambda-2)^2}\left(\frac{B^{1/2}r}{r_0}\right)^{\lambda}\right.$$
$$\left.+\frac{\lambda^2}{(\lambda-2)^2}\left(\frac{B^{1/2}r}{r_0}\right)^2\right] + \dot r_0^2 + r_0^2\dot\theta_0^2\left[1-2B\frac{\lambda+1}{\lambda-1}\right] \tag{13b}$$

We find that from Eqs. (4) and (13) $\dot\theta$ will approach $[\lambda/(\lambda-2)]\dot\theta_B$, no matter if B is greater than 1 or not, as shown in Fig. 3. The larger the bias is, the more the closing speed will be reduced during intercept, as depicted in Fig. 4. The final rate of range to go can be obtained as

$$\dot r_f = \dot r_0\sqrt{1-\frac{1}{A^2}\left[2B\frac{\lambda+1}{\lambda-1}-1\right]} \tag{14}$$

Thus, the following constraint must be satisfied for effective intercept of target:

$$A^2 > 2B\frac{\lambda+1}{\lambda-1}-1 \tag{15}$$

Therefore, a larger value of A^2 is required for a larger bias, and the cumulative velocity increment ΔV required for the missile during intercept is

$$\Delta V = \left|\int_{r_0}^{0}\frac{a_c}{\dot r}\,\mathrm{d}r\right| = |r_1\dot\theta_1|\lambda\int_0^1\left[\frac{2}{\lambda-2}\right.$$
$$\left.-\frac{2}{\lambda-2}\left(\frac{r}{r_1}\right)^{\lambda-2}\right]\mathrm{d}\times\left(\frac{r}{r_1}\right) = |r_0\dot\theta_0|B^{1/2}\frac{2\lambda}{\lambda-1} \tag{16}$$

We find that here the bias affects the cumulative velocity with a factor $B^{1/2}$ when $B>1$.

III. Solution for a Maneuvering Target

For the case of a maneuvering target, we assume that the target maneuver is also in the direction normal to the LOS, same as the commanded acceleration, and its magnitude is

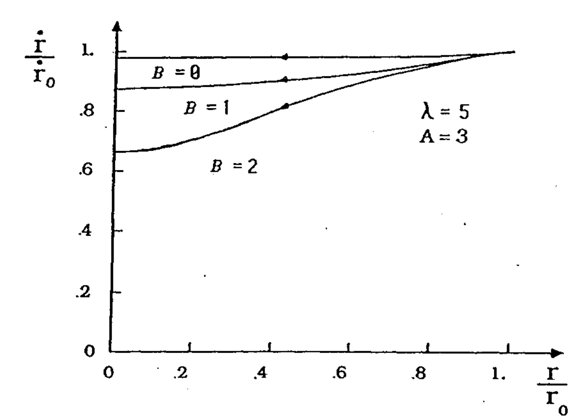

Fig. 4 Different values of B: $\dot r$ vs r.

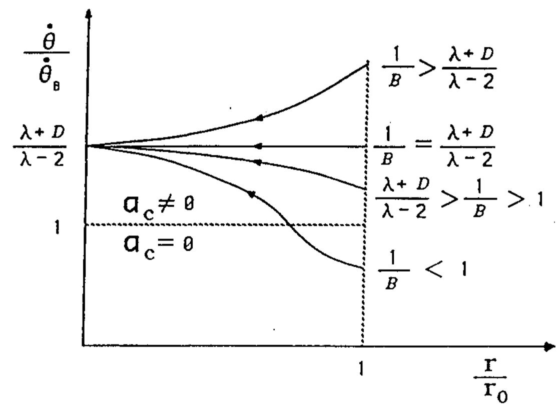

Fig. 5 Biased PN with maneuvering target: $\dot{\theta}$ vs r.

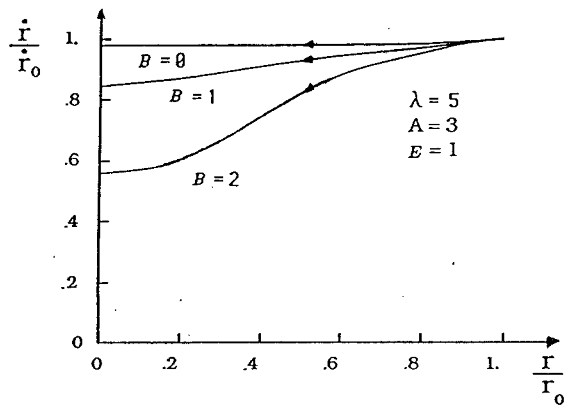

Fig. 6 Different values of B with maneuvering target: $\dot{r}$ vs r.

proportional to the closing rate for effective escape during intercept period. A larger target maneuver is acquired for larger closing rate and no target maneuver is required when closing rate approaches zero. Then we have the equations of relative motion between missile and target:

$$\ddot{r} - r\dot{\theta}^2 = 0 \tag{17a}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \lambda\dot{r}(\dot{\theta} - \dot{\theta}_B) - a_T \quad \text{if} \quad \frac{\dot{\theta}}{\dot{\theta}_B} \geq 1 \tag{17b}$$

$$= -a_T \quad \text{if} \quad \frac{\dot{\theta}}{\dot{\theta}_B} < 1 \tag{17c}$$

with $a_T = -C\dot{r}$ (C is a constant). In the case of $B \leq 1$, the solutions of Eqs. (17) can be derived as (see Ref. 4):

$$\dot{\theta} = \dot{\theta}_0\left\{\left(\frac{r}{r_0}\right)^{\lambda-2} + \frac{\lambda + D}{\lambda - 2}B\left[1 - \left(\frac{r}{r_0}\right)^{\lambda-2}\right]\right\} \tag{18a}$$

$$\dot{r}^2 = r_0^2\dot{\theta}_0^2\left\{\frac{1}{\lambda-1}\left[1 - \frac{\lambda B + E}{\lambda - 2}\right]^2\left(\frac{r}{r_0}\right)^{2\lambda-2} + \frac{4(\lambda B + E)}{\lambda(\lambda-2)}\left[1 - \frac{\lambda B + E}{\lambda - 2}\right]\left(\frac{r}{r_0}\right)^{\lambda} + \frac{(\lambda B + E)^2}{(\lambda-2)^2}\left(\frac{r}{r_0}\right)^2\right\} + \dot{r}_0^2 - \frac{r_0^2\dot{\theta}_0^2}{\lambda - 1}\left\{1 + \frac{1}{\lambda}[(1 + \lambda B + E)^2 - 1]\right\} \tag{18b}$$

where $E = -C/\dot{\theta}_0$ and $D = -C/\dot{\theta}_B$ (i.e., $E = DB$). Here, positive escape factor E is considered, which intends to diverge the LOS rate between missile and target. We find that from Eqs. (18) $\dot{\theta}$ approaches $[(\lambda + D)/(\lambda - 2)]\dot{\theta}_B$ at intercept, with an additional term $[D/(\lambda - 2)]\dot{\theta}_B$ compared to the case of nonmaneuvering target. Figure 5 shows the typical variation between $\dot{\theta}$ and r under the effects of both bias and target maneuver. The typical variation between $\dot{r}$ and r is depicted in Fig. 6. The final closing rate at intercept can be obtained as

$$\dot{r}_f = \dot{r}_0\sqrt{1 - \frac{1 + (1/\lambda)[(1 + \lambda B + E)^2 - 1]}{A^2(\lambda - 1)}} \tag{19}$$

and the final commanded acceleration approaches $[\lambda/(\lambda - 2)](2\dot{r}_f\dot{\theta}_B + a_T)$. Thus, the following constraint must be satisfied for effective intercept of target with $B \leq 1$:

$$A^2 > \frac{1}{\lambda - 1}\left\{1 + \frac{1}{\lambda}[(1 + \lambda B + E)^2 - 1]\right\} \tag{20}$$

In this case, the zero effort miss M can be derived as (see Appendix B)

$$M = rx \tag{21}$$

in which x is the solution of

$$\left(1 + \frac{E}{2}\right)^2 x^{-2} + 2E\left(1 + \frac{E}{2}\right)\ell n\, x - \frac{E^2}{4}x^2 = \frac{\dot{r}^2 + r^2\dot{\theta}^2(1 + E)}{r^2\dot{\theta}^2} \tag{22}$$

The cumulative velocity increment ΔV can be obtained as follows:

$$\Delta V = \left|\int_{r_0}^{0}\frac{a_c}{\dot{r}}\,\mathrm{d}r\right| = |r_0\dot{\theta}_0|\lambda\int_0^1\left[\frac{2B + E}{\lambda - 2} + \left(1 - \frac{\lambda B + E}{\lambda - 2}\right)\times\left(\frac{r}{r_0}\right)^{\lambda-2}\right]\mathrm{d}\left(\frac{r}{r_0}\right) = |r_0\dot{\theta}_0|\frac{\lambda}{\lambda - 1}(1 + B + E)$$
$$= \frac{\lambda}{\lambda - 1}r_0[|\dot{\theta}_0| + |\dot{\theta}_B| + |C|] \tag{23}$$

Next, the case of $B > 1$ is considered. In this case, $\dot{\theta}$ diverges first with $a_c = 0$ to approach $\dot{\theta}_B$, where the corresponding value of r is r_1, and then diverges continuously to $[(\lambda + D)/(\lambda - 2)]\dot{\theta}_B$ with $a_c \neq 0$ until intercept.

1) When $r > r_1$: Because a_c is equal to zero in this area, its solution can be obtained from Eqs. (17a) and (17c) as

$$\dot{\theta} = \dot{\theta}_0\left\{\left(\frac{r}{r_0}\right)^{-2} - \frac{E}{2}\left[1 - \left(\frac{r}{r_0}\right)^{-2}\right]\right\} \tag{24a}$$

$$\dot{r}^2 = r_0^2\dot{\theta}_0^2\left[-\left(1 + \frac{E}{2}\right)^2\left(\frac{r}{r_0}\right)^{-2} - 2E\left(1 + \frac{E}{2}\right)\ell n\left(\frac{r}{r_0}\right) + \frac{E^2}{4}\left(\frac{r}{r_0}\right)^2\right] + \dot{r}_0^2 + r_0^2\dot{\theta}_0^2(1 + E) \tag{24b}$$

Let $\dot{r}_1$ be the value of $\dot{r}$ at the time when $\dot{\theta}$ reaches $\dot{\theta}_B$, then we have

$$\dot{\theta}_1 = \dot{\theta}_B = \dot{\theta}_0 B \tag{25a}$$

$$r_1 = r_0\sqrt{\frac{2 + E}{2B + E}} \tag{25b}$$

$$\dot{r}_1 = \dot{r}_0\sqrt{1 + \frac{1}{A^2}\left[1 + E - \frac{1}{4}(2 + E)(2B + E) - \frac{1}{2}E(2 + E)\,\ell n\frac{2 + E}{2B + E} + \frac{E^2}{4}\frac{2 + E}{2B + E}\right]} \tag{25c}$$

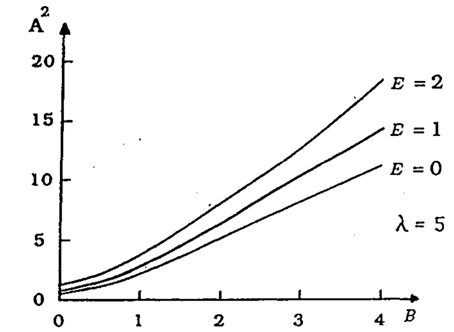

Fig. 7 Capture boundary of biased PN with maneuvering target.

After r reaches r_1, it decreases continuously with $a_c \neq 0$ as follows.

2) When $r \leq r_1$: The solution here is the same as obtained in Eqs. (18). With the initial condition $r = r_1$, $\dot\theta = \dot\theta_1 = \dot\theta_B$, and $r = r_1$, we have

$$\dot\theta = \dot\theta_1\left\{\left(\frac{r}{r_1}\right)^{\lambda-2} + \frac{\lambda+D}{\lambda-2}\left[1-\left(\frac{r}{r_1}\right)^{\lambda-2}\right]\right\}$$

$$= \dot\theta_0 B\left[\frac{\lambda+D}{\lambda-2} - \frac{2+D}{\lambda-2}\left(\sqrt{\frac{2B+E}{2+E}}\,\frac{r}{r_0}\right)^{\lambda-2}\right] \tag{26a}$$

$$\dot r^2 = r_1^2\dot\theta_1^2\left\{\frac{1}{\lambda-1}\left[1-\frac{\lambda+D}{\lambda-2}\right]^2\left(\frac{r}{r_1}\right)^{2\lambda-2} + \frac{4(\lambda+D)}{\lambda(\lambda-2)}\left[1-\frac{\lambda+D}{\lambda-2}\right]\left(\frac{r}{r_1}\right)^{\lambda} + \left(\frac{\lambda+D}{\lambda-2}\right)^2\left(\frac{r}{r_1}\right)^2\right\}$$

$$+\dot r_1^2 - \frac{r_1^2\dot\theta_1^2}{\lambda-1}\left\{1+\frac{1}{\lambda}\left[(1+\lambda+D)^2-1\right]\right\}$$

$$= r_0^2\dot\theta_0^2B^2\frac{2+E}{2B+E}\left\{\frac{1}{\lambda-1}\left(\frac{2+D}{\lambda-2}\right)^2\left(\sqrt{\frac{2B+E}{2+E}}\,\frac{r}{r_0}\right)^{2\lambda-2} - \frac{4(\lambda+D)(2+D)}{\lambda(\lambda-2)^2}\left(\sqrt{\frac{2B+E}{2+E}}\,\frac{r}{r_0}\right)^{\lambda} + \left(\frac{\lambda+D}{\lambda-2}\right)^2\left(\sqrt{\frac{2B+E}{2+E}}\,\frac{r}{r_0}\right)^2\right\} + \dot r_f^2 \tag{26b}$$

where $\dot r_f$ is the final range rate at intercept, which can be obtained as

$$\dot r_f = \dot r_0\left(1 - \frac{1}{A^2}\left\{\frac{B^2(2+E)}{(\lambda-1)(2B+E)}\left[1+\frac{1}{\lambda}\left(\lambda+\frac{E}{B}\right)^2 + \frac{2}{\lambda}\left(\lambda+\frac{E}{B}\right)\right] - (1+E) + \frac{1}{4}(2+E)(2B+E) - \frac{E^2}{4}\left(\frac{2+E}{2B+E}\right) + \frac{E}{2}(2+E)\ln\frac{2+E}{2B+E}\right\}\right)^{1/2} \tag{27}$$

Thus, the capture criterion in this case is

$$A^2 > \frac{B^2(2+E)}{(\lambda-1)(2B+E)}\left[1+\frac{1}{\lambda}\left(\lambda+\frac{E}{B}\right)^2 + \frac{2}{\lambda}\left(\lambda+\frac{E}{B}\right)\right] - (1+E) + \frac{1}{4}(2+E)(2B+E) - \frac{E^2}{4}\left(\frac{2+E}{2B+E}\right) + \frac{E}{2}(2+E)\ln\frac{2+E}{2B+E} \tag{28}$$

With a typical value of λ, the capture criterion A^2 as a function of the bias factor B and target escape factor E is shown in

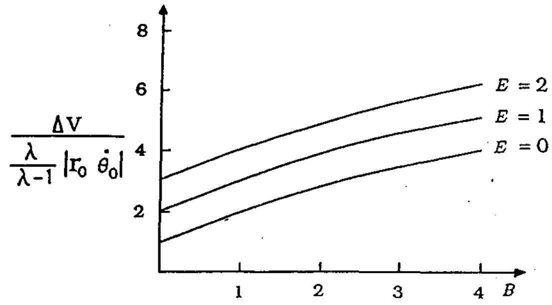

Fig. 8 ΔV vs B with maneuvering target.

Fig. 7. For a given value of E, the area above the curve is the capture area and the area below the curve is the noncapture area. It is obvious that the capture area becomes smaller for a larger value of the target escape factor. In summary, we need higher A^2 when B or E is higher so that the intercept can be guaranteed, as expected.

The cumulative velocity increment ΔV in this case can be obtained as

$$\Delta V = \left|\int_{r_1}^{0}\frac{a_c}{\dot r}\,dr\right| = |r_1\dot\theta_1|\lambda\frac{2+D}{\lambda-2}\int_0^1(1-x^{\lambda-2})\,dx$$

$$= |r_0\dot\theta_0|\frac{\lambda}{\lambda-1}\sqrt{(2B+E)(2+E)} \tag{29}$$

We find that, if E is equal to 0, the results obtained for the case of a maneuvering target will be the same as those obtained for the case of a nonmaneuvering target.

The effect of bias with target maneuver on cumulative velocity increment is depicted in Fig. 8. Again, the required cumulative velocity increment is higher for a higher value of B and E.

IV. Discussion

From this study, we found that the LOS rate will not approach zero in biased proportional navigation. It approaches $[\lambda/(\lambda-2)]\dot\theta_B$ for a nonmaneuvering target, and approaches $[(\lambda+D)/(\lambda-2)]\dot\theta_B$ for a maneuvering target. The bias will also decrease the capture capability and increase the cumulative velocity required for intercept of target. The more the bias is, the less the capture area is and the more the cumulative velocity increment is required. When $B<1$, the cumulative velocity increment is a linear combination of the function of B (due to bias) and the function of E (due to target maneuver), respectively, as shown in Eqs. (23). When $B>1$, then it will become a nonlinear function of bias and target maneuver, as shown in Eqs. (29).

V. Conclusions

An analytic study of the biased proportional navigation is presented in this paper. Both maneuvering and nonmaneuvering targets are considered. In this scheme, no matter what value the bias is, the line-of-sight rate approaches a nonzero value that is beyond the dead-zone of the system. Intercept can be still achieved with a cost of additional energy, and the capture area will be decreased due to the effect of bias.

Appendix A: Zero Effort Miss for a Nonmaneuvering Target

For a nonmaneuvering target, the relative motion between missile and target can be written as

$$\ddot r - r\dot\theta^2 = 0 \tag{A1a}$$

$$r\ddot\theta + 2\dot r\dot\theta = 0 \tag{A1b}$$

The solutions of Eqs. (A1) are

$$\dot{\theta} = \frac{c_1}{r^2} = \frac{r_0^2\dot{\theta}_0}{r^2} \tag{A2a}$$

$$\dot{r}^2 = -\frac{(r_0^2\dot{\theta}_0)^2}{r^2} + c_2 = -\frac{(r_0^2\dot{\theta}_0)^2}{r^2} + \dot{r}_0^2 + r_0^2\dot{\theta}_0^2 \tag{A2b}$$

Then the zero effort miss M is equal to the minimum value of r that occurred during the relative motion governed by Eqs. (A2), at which $\dot{r}$ is equal to zero, i.e.,

$$M = r_{\min} = \sqrt{\frac{(r_0^2\dot{\theta}_0)^2}{\dot{r}_0^2 + r_0^2\dot{\theta}_0^2}} = r_0\sqrt{\frac{r_0^2\dot{\theta}_0^2}{\dot{r}_0^2 + r_0^2\dot{\theta}_0^2}}$$

Thus, the zero effort miss for any instant during intercept period can be described as

$$M = r\sqrt{\frac{v_\theta^2}{v_r^2 + v_\theta^2}} \tag{A3}$$

Appendix B: Zero Effort Miss for a Maneuvering Target

For a maneuvering target, the relative motion between missile and target can be written as

$$\ddot{r} - r\dot{\theta}^2 = 0 \tag{B1a}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -a_T = C\dot{r} \tag{B1b}$$

The solutions of Eqs. (B1) are

$$\dot{\theta} = \dot{\theta}_0\left[\left(1 + \frac{E}{2}\right)\left(\frac{r}{r_0}\right)^{-2} - \frac{E}{2}\right] \tag{B2a}$$

$$\dot{r}^2 = r_0^2\dot{\theta}_0^2\left[-\left(1 + \frac{E}{2}\right)^2\left(\frac{r}{r_0}\right)^{-2} - 2E\left(1 + \frac{E}{2}\right)\ln\frac{r}{r_0} + \frac{E^2}{4}\left(\frac{r}{r_0}\right)^2\right] + \dot{r}_0^2 + r_0^2\dot{\theta}_0^2(1 + E) \tag{B2b}$$

in which $E = -C/\dot{\theta}_0$. Then the zero effort miss can be derived as

$$M = r_0 x \tag{B3}$$

where x satisfies

$$\left(1 + \frac{E}{2}\right)^2 x^{-2} + 2E\left(1 + \frac{E}{2}\right)\ln x - \frac{E^2}{4}x^2 = \frac{\dot{r}_0^2 + r_0^2\dot{\theta}_0^2(1+E)}{r_0^2\dot{\theta}_0^2} \tag{B4}$$

Thus, the zero effort miss for any instant during intercept period is

$$M = rx \tag{B5}$$

where x satisfies

$$\left(1 + \frac{E}{2}\right)^2 x^{-2} + 2E\left(1 + \frac{E}{2}\right)\ln x - \frac{E^2}{4}x^2 = \frac{v_r^2 + v_\theta^2(1+E)}{v_\theta^2} \tag{B6}$$

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